Self-organized percolation

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We investigate a growth mechanism for percolation clusters where self-organization leads the system to criticality. By controlling the number of sites or bonds in the growth front of the aggregate, the system is spontaneously driven to a stationary state that corresponds to the percolation threshold of the lattice topology and percolation process used in the simulation (site or bond percolation). This self-tuning behavior around a critical state is then discussed with reference to the concept of self-organized criticality. We also suggest that the generalization of this approach to other controlling rules and lattice geometries could explain the occurrence of percolation structures in some physical and nonphysical disordered systems.
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Percolation is a successful model for many phenomena in nature [1]. In its original version, percolation is a *static* geometric model in the sense that we have to simultaneously populate all sites or bonds of a lattice with a prescribed probability p. For small values of p, only finite clusters will be present. By increasing p, one can find the threshold or critical value p_c for which an infinitely connected object is generated.

Extensions of the percolation model have been developed to describe dynamical phenomena. In the model of Leath [2], all sites are visited and populated with a *fixed* probability p during the growth process. Obviously, this requires a trial and error scheme to determine the critical probability, p_c. The advantage is that, since $p=p_c$ even at the local level, critical clusters generated with this algorithm have morphological characteristics that are identical to standard percolation ones.

In the *invasion percolation model* [3], a random number uniformly distributed in [0,1] is assigned to each site or bond, and the cluster growth from an active element takes place through the occupation of the neighbor position that has the smallest probability value. In spite of the fact that the *local probability distribution* of the occupied element in invasion percolation is very broad, the clusters generated from the invasion and standard (static) percolation models have approximately the same fractal dimension, $d_f \approx 1.89$.

The concept of self-organized criticality (SOC) has been introduced by Bak, Tang, and Wiensenfeld [4] as a possible explanation for the tendency of large and complex systems to drive themselves to critical states. Accordingly, genuine SOC systems would be expected to display power law correlations in time and space. Additionally, the critical state would have to be achieved without the need of imposing a fine tuning mechanism to a given parameter. In summary, the two essential features for the existence of a SOC state are *self-organization* and *self-similarity*. However, the occurrence of both properties in the same system to exhibit SOC behavior still represents a very controversial issue. Moreover, it has been questioned if a mean-field description for a SOC system could provide some insight that could explain or somehow justify the phenomenon of spontaneous organization towards a critical state. Zapperi, Lauritsen, and Stanley [5] solved this problem introducing a new mean-field model, the self-organizing branching process (SOBP). Their model is based on a sandpile paradigm in which open boundary conditions are explicitly incorporated to allow for dissipation. Local dynamical rules are conveniently coupled to a global controlling mechanism and, as a result, the system organizes itself into a stationary state described by a critical branching process.

In the present communication, we find that simple controlling rules can lead to a *self-organized percolation* (SOP) process. Moreover, this state is reached and sustained close to the percolation threshold without fine tuning any parameter in the system. In our model, we start with several occupied sites at one side of a $L \times L$ square lattice. At the initial time ($t=0$), the available nearest-neighbors of these seeds are identified and considered occupied if the uniform random numbers $0 \leqslant r \leqslant 1$ assigned to them are less than a given probability p. The sites occupied at time t in the lattice are called the active sites of the growing process. However, in contrast with the method developed by Leath [2], where the probability p is fixed during the growth of the *spanning cluster*, here we allow it to vary in order to compensate for the increase or decrease in the number of active sites, $N(t)$, in the growth front. The basic idea is to ensure that the cluster will never stop growing and also that the number of active sites will never increase exponentially in time. For simplicity, we adopted the following first order controlling mechanism

$$p(t+1)=p(t)+k[N_T-N(t)], \tag{1}$$

where k is a kinetic coefficient and N_T is a threshold parameter. The probability is obviously limited to the range $0 \leqslant p \leqslant 1$, so that we have to impose $p(t+1)=0$ and $p(t+1)=1$ if the recurrence relationship Eq. (1) gives nonphysical values for $p(t+1)$ less than 0 and greater than 1, respectively. As in the traditional algorithm for growth of percola-

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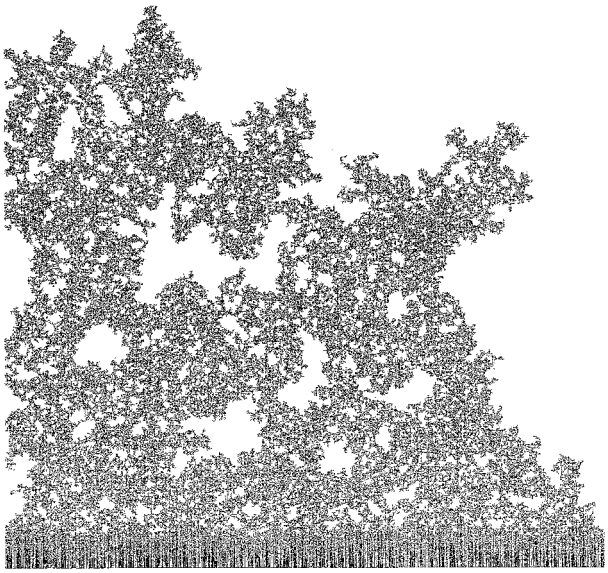

FIG. 1. Typical realization of a cluster grown under the bond percolation scheme and driven by the controlling rule Eq. (1). In this particular simulation, 100 seeds have been randomly allocated in the bottom line of a 1000×1000 square lattice. The initial growth probability is $p_0=1$, the kinetic coefficient is $k=10^{-5}$ and the threshold for the number of active elements is $N_T=200$.

tion clusters [1], if a selected perimeter site is not occupied at a given time step, it will remain unnocupied forever. Open boundary conditions have been adopted at the sides of the lattice that are orthogonal to the boundary line where the seeds have been placed. The generalization of the whole algorithm for bond percolation growth is straightforward.

A typical realization of a cluster grown with our model is shown in Fig. 1. As depicted, the cluster density follows the variation in the parameter p, initially growing as a compact object at the lower part of the lattice and changing gradually to a ramified structure which resembles more closely a percolation geometry. The time evolution of the growth probabilty p is shown in Fig. 2(a) for three different values of p_0. After a transient period and independently of p_0, the systems reaches a stationary state characterized by an average value of the growth probability $\langle p\rangle\approx0.59$, which closely recovers the percolation threshold p_c obtained with standard simulations of site percolation clusters [1]. In Fig. 2(b), the results of simulations performed for the bond percolation case also indicate that, after a transient interval and independently of p_0, the growth probability p becomes stationary and starts executing small amplitude oscillations around an average value $\langle p\rangle\approx0.5$. This result is consistent with the critical point reported in literature for bond percolation [1]. To confirm the hypothesis of self-organization in a critical state, extensive simulations have been carried out with distinct kinetic coefficients k, threshold values N_T as well as applying different controlling rules to systematically change the variable p. In all cases, we observed the same self-organized behavior with the system being automatically tuned to the corresponding site or bond percolation threshold.

The restoring mechanism given by Eq. (1) regulates the probability p in such a way that the continuity of the growth process is kept at a minimum rate. As expected, the results of our simulations indicate that the threshold value N_T simply

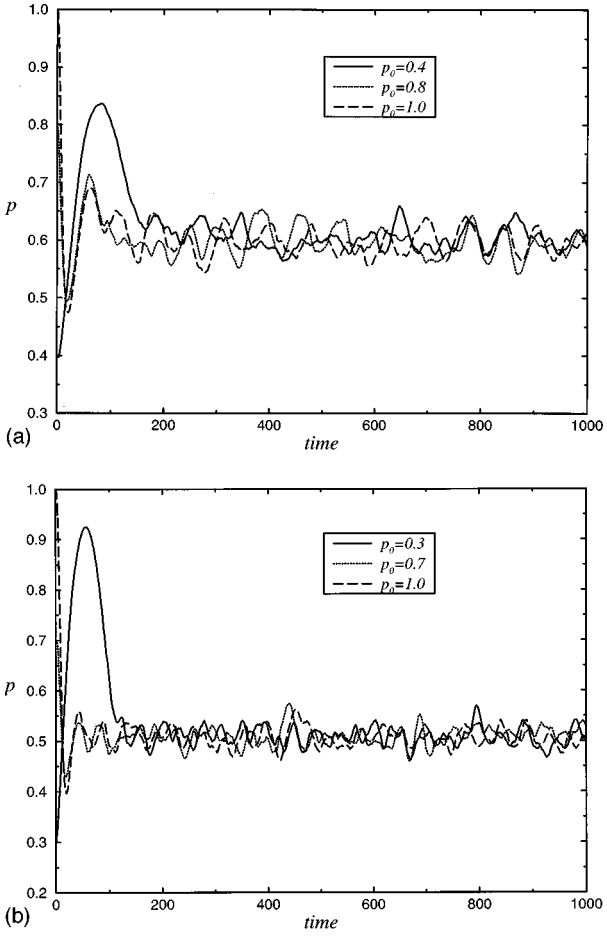

FIG. 2. (a) Plot of the probability of occupation in the growth front p as a function of time for the site percolation case in a 2000×2000 square lattice. The controlling rule used is Eq. (1) ($k=10^{-4}$ and $N_T=200$). After a short transient and independently of the initial conditions p_0, $p(t)$ achieves a stationary state, $\langle p\rangle\approx0.59$, and fluctuates around it with short range correlations. (b) The same as in (a), but now for the bond percolation case. Again, independently of p_0 and after a short transient interval, $p(t)$ executes small amplitude oscillations around a critical state, $\langle p\rangle\approx0.5$.

sets up the average value around which the number of active sites N fluctuates in time after the system achieves stationarity (see Fig. 3). Evidently, N_T cannot be too small, otherwise fluctuations in the process could eventually kill the remaining active tips in the growth front. On the other hand, large values of N_T would require an excessively long transient period (and so large lattice sizes) for the SOP system to reach the stationary regime.

The simulation results displayed in Fig. 3 clearly indicate that the parameter k has a strong influence on the oscillatory behavior of the function $N(t)$. Although the time series shown in Fig. 3 do not follow a periodic pattern, the high frequency fluctuations in $N(t)$ appear to be associated with large values of k. This effect can be justified through the following mean-field representation of the system:

$$N(t+1)=3p(t)[1-s(t)]N(t)\,, \tag{2}$$

where the deactivation probability $s(t)$, $0\leqslant s(t)\leqslant1$, accounts for exclusion effects during the SOP growth process.

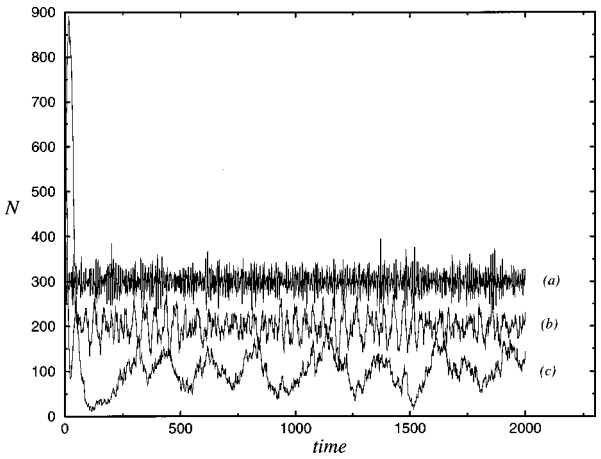

FIG. 3. Dependence on time of the number of active sites $N(t)$ calculated at distinct threshold positions N_T for different values of the parameter k: (a) $N_T=300$ and $k=10^{-3}$; (b) $N_T=200$ and $k=10^{-4}$; (c) $N_T=100$ and $k=10^{-5}$.

These self-avoiding mechanisms are induced by spatial correlations in the lattice and tend to reduce the current number of active sites in the system. Equations (1) and (2) constitute the basis of the SOP dynamics. Taking the continuum limits of them and assuming that the probability s is approximately constant for a given lattice topology, after some manipulations, the following differential equation in $N(t)$ is obtained:

$$\frac{d^2N}{dt^2}=\frac{1}{N}\left(\frac{dN}{dt}\right)^2+3k(1-s)N_TN-3k(1-s)N^2. \quad (3)$$

The last term on the right-hand side of Eq. (3) indicates that the occurrence of high frequency oscillations in $N(t)$ should be directly, but in a nonlinear fashion, associated with large values of the coefficient k. This behavior is in agreement with the results shown in Fig. 3.

It is interesting to investigate different forms of the controlling mechanism acting on percolation growth. For example, we can stipulate that the threshold N_T increases with time according to a power law type of function, $N_T(t)=at^{\alpha}$. Instead of Eq. (1), the following recurrence relationship should then be utilized in the SOP model:

$$p(t+1)=p(t)+k[at^{\alpha}-N(t)], \quad (4)$$

where $a>0$ and $\alpha>0$ are arbitrary constants. Actually, this feedback rule could simulate the situation of growth control being exerted by an underlying coexisting process whose dynamics is expressed here as $N_T(t)$. Remarkably, our results indicate that, within a certain range of α values, $0\leqslant\alpha<1$, clusters grown under the controlling mechanism Eq. (4) are still self-organized around critical states corresponding to the specific percolation thresholds of lattice topologies and aggregation rules. In this case, while the function $N(t)$ closely follows the time evolution of the variable threshold $N_T(t)$ [see Fig. 4(a)], the probability $p(t)$ rapidly achieves a stationary state around a fixed point $\langle p\rangle\approx0.59$ [see Fig. 4(b)], which is equal to the threshold value for site percolation in a square lattice [1].

The SOP process can also be compared to other methods to determine critical coordinates of phase transitions. In the

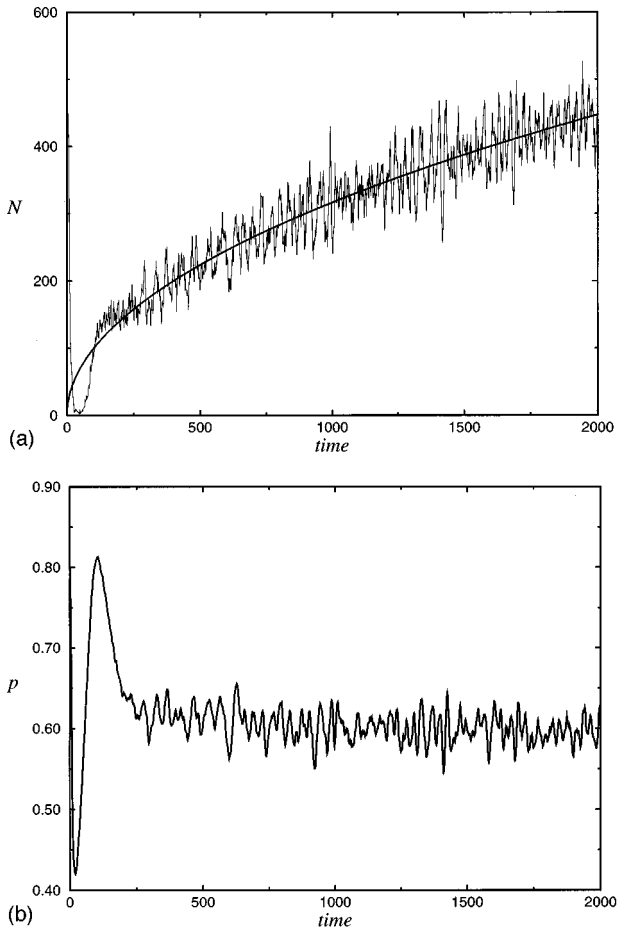

FIG. 4. (a) Dependence on time of the number of active sites $N(t)$ for a site percolation cluster growing under the controlling rule Eq. (4) ($k=10^{-4}$ and $p_0=0.8$). The thick line corresponds to $N_T(t)=10t^{1/2}$. (b) The same as in (a), but now for the probability p of occupation in the growth front. For this particular simulation, the growth process started from 100 seeds which have been randomly allocated in the bottom line of a 2000×2000 square lattice.

gradient method [6], a gradient is imposed to the order parameter against which the cluster has to grow. At each time step, as the cluster frontier (as a whole) advances in a given direction, the value of p is reduced of a constant amount, obeying a linear relationship. This corresponds to a gradient of p in time and also in space. When p reaches p_c, the object starts to loose its global connectivity. Thus, the average boundary of the connected cluster marks the value of p_c. The drawback of the method is that the gradient is fixed and only the external surface of the percolation cluster is at criticality. Its accuracy depends on the possibility of repeating the experiment and confining the set of p_c estimates to smaller and smaller intervals. Compared to the gradient method, our model is analogous to have, at each time step, a variable gradient of p that does not work against growth all the time, but always drives the system back to criticality. This gradient is not fixed, but depends on the deviation from the critical point. So if the system is far from p_c, the gradient is big, whereas if p is close to p_c, the gradient is small. Under these rules, the system will approach the critical state in a fast way and will remain close to p_c, providing an easy and precise technique to determine this value in a single run.

Before concluding, we point out that the intrinsic dynamics involved in the SOP process is also present in other growth systems where some forms of self-organization and criticality are simultaneously observed. Genuine SOC states have also been detected in the branched polymer growth model (BPGM) [7,8]. In a recent study [9], numerical simulations have been performed with the BPGM to demonstrate how a simple and feasible feedback mechanism like Eq. (1) could systematically drive the aggregation process to the vicinity of the transition line between finite and infinite regimes of growth.

Summarizing, in this paper we proposed a growth mechanism for percolation clusters, the SOP process, in which self-organization and criticality are ingredients of the same dynamics. We demonstrated that a SOC state can be promptly and spontaneously achieved by coupling the Leath algorithm [2] to global controlling rules that simply regulate the growth rate of a percolation aggregate. Under this framework, we also pointed out that the mechanism introduced here should be flexible to simulate the growth of other physical and non-physical processes. For instance, in the context of biological systems, recent studies [10,11] have demonstrated that while blood capillary networks present in normal tissues are fully connected or compact (Euclidian) structures, the process of vascular network formation (angiogeneses) in cancer tumors display percolationlike scaling. If tumor angiogeneses is limited by cancer growth, it should be possible to understand the morphological characteristics of tumor vascular networks in the framework of an SOP process.

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